Qualitative theory of differential equations and structural stability

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1. Introduction.

The qualitative or geometrical theory of ordinary differential equations -- as opposed or better complementary to the quantitative, numerical or analytical theory -- was considered for the first time, in 1881, by Poincaré in his famous mémoire "Sur les courbes définies par une équation différentielle" [13]. This paper was so ahead of its time in scope and outlook that two or three decades had to pass before it began to be assimilated and progress beyond it was possible. Even today browsing through the memoire may be rewarding for the working mathematician for here and there he is apt to find at some dark corner -- and there are many in this vast edifice-- a meaningful problem or an idea worth polishing.

Till his death in 1912 Poincare kept a keen interest in some aspects of this subject, especially that part related to Celestial Mechanics [14]. That work was carried on by G. D. Birkhoff [5] who wrote extensively in this area. It is well known how in 1913 the young Birkhoff acquired sudden fame by solving a question on fixed points proposed

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by Poincaré in his last paper and which appeared in his investigations on the three body problem.

In the hands of Poincare, Birkhoff and Liapunov many outstanding qualitative results about differential equations were obtained and many basic concepts were established.

However only recently has the qualitative theory been put on a solid basis with the formulation by Smale [17] of the fundamental problem of the theory as a fairly precise mathematical problem. Instrumental in this formulation was the concept of structural stability, in the sense that the role played by structurally stable systems on the disc B^2 suggested what to look for in the general case.

These recent developments of the qualitative theory were very much influenced by the methods and ideas of Differential Topology, specially Thom's transversality [23]. There seems to be little doubt however that more than tranversality is involved when we have to deal with the all important recurrent trajectories, as in the problem of the closing Lemma mentioned below.

In this survey we first give an account on how the formulation of the fundamental problem fits with previous developments and of its present status. Then we announce a few results and make several comments and conjectures, all related to the place of structural stability in the general picture of the qualitative theory. In particular we consider the role of first integrals and remark that a theorem of Thom about them follows easily from a general density theorem.

At present it is not known how important the concept of structural stability will turn out to be for the qualitative theory. A non density example of Smale [18] shows that it is not going to be all important for the fundamental problem in dimension $n \geqq 4$.

In a much more general setting than the strictly mathematical one considered here Thom is writing a highly original and daring book on structural stability. Undisturbed by Smale's example he sees structural stability, broadly understood as the preservation of qualitative features under small perturbation, as an axiom to be put on any model of a natural process, a kind of morphological substractum of natural law. Something that has to do with the cresting of sea waves, with the way liquids mix, with biological order and growth, and so forth.

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2. Qualitative theory: the fundamental problem.

We are concerned with ordinary differential equations, also called vector fields or dynamical systems defined on a finite dimensional manifold M^n. Poincare was the first to consider differential equations defined in manifolds other than Euclidean space and to prove theorems of a global and qualitative nature about them. He never tried to make precise what one should understand by "qualitative".

For our purpose here it is relevant to mention that the idea of "genericity", that one should look for situations that are present for most values of the "coeficients" is mentioned many times in Poincare's memoire. In particular the concept of generic singularity (for which the eigenvalues of the Jacobian matrix have non-zero real parts) and the similar one of generic closed orbit are formulated by him. He was also aware of what today we call the stable and unstable manifolds associated to these elements and even of the much more subtle concept of homoclinic point [14, Vol. 3, 20].

Below we indicate some further steps that were taken by different authors at different times and which resulted in the formulation mentioned above, of the fundamental problem of the qualitative theory of differential equations.

(2.1) In 1924 H. Kneser [7] considered certain types of differential equations on the torus T^2 and said that two of them X and Y were equivalent, $X \sim Y$, if a homeomorphism $h: T^2 \to T^2$ could be found such that h maps trajectories of X onto trajectories of Y. He then classified these differential equations, i.e., exhibited the corresponding equivalence classes. To the author's knowledge this was the first step towards a clarification of what one should understand by qualitative behavior.

(2.2) In 1937 A. Andronov and L. Pontrjagin [1] considered differential equations on the ball B^2 and said that a system X is "rough" if by perturbing it slightly in the C^1-sense then one gets a system $Y \sim X$ (in the above sense) and the corresponding homeomorphism can be made arbitrarily small by taking Y close enough to X. Then they gave a set of conditions as being necessary and sufficient for X to be rough. These conditions turn out to exclude complicated behavior for the trajectories of X; we will come again to this point later. Questions of this type in many special instances were at that time considered by soviet engineers.

It seems that Lefschetz [2,8] was the first to realize that here was an important mathematical concept that required further investigation. Translating rough by the much better sounding "structurally stable" he exhibited the true meaning of the new concept namely a fusion of the two concepts of stability and qualitative behavior in the sense of topological equivalence.

(2.3) In 1940 W. Kaplan [6] in his thesis gave a complete classification of all equivalence (in the above sense) classes of differential equations on R^2 having no singularities, reducing this complex problem (there are non countable many equivalence classes) to a purely algebraic and combinationial question. To this day Kaplan's classification seems to be the deepest result of this type. One should mention that in L. Markus thesis [9] Kaplan's methods were extended to certain types of equations on R^2 with singularities.

(2.4) In 1959 the author [10] considered the situation treated by Andronov and Pontriagin i.e., structural stability on B^2 (the case with S^2 is the same) and showed that by making a metric space $\mathfrak{X}$ out of the set of all differential equations (with the C^1-topology) the ones which are structurally stable Σ constitute a set which is open and dense in $\mathfrak{X}$ and besides they exhibit very simple qualitative features. In other words "almost all" differential equations in S^2 are very simple as far as the topological behavior of trajectories go.

(2.5) The next important step was taken by Smale [17]. Let $\mathfrak{X}$ be the space of all vector fields on a compact differentiable manifold M^n with the C^r-topology $r \geqq 1$.

The fundamental problem of the qualitative theory of differential equations on M^n is: to exhibit in $\mathfrak{X}$ a dense subset such that the features of the corresponding systems are simple enough as to make them amenable to classification.

There are two points that need comment here. First that one should not try to classify all systems on $\mathfrak{X}$ because this is too difficult. One is easily convinced of that by considering the vector fields on S^1: to classify these vector fields amounts to classify the closed sets on S^1. The second point is that one should not necessarily insist on a classification with respect to the equivalence relation $\sim$ of a homeomorphism mapping trajectories onto trajectories. If one insists in classifying a dense family one might settle for something weaker than $\sim$.

The point then in Smale's formulation is the combination of the simultaneous requirements of "genericity", to be understood in the precise sense of density in $\mathfrak{X}$ and "simplicity" to be understood in the somewhat vague sense of features simple enough as to lead to a classification.

No small credit is due to Smale for having realized immediately that what was significant in the S^2 case was the simultaneous presence of these two features and that the added feature of structural stability, enticing and important on its own as it might be, was not necessarily relevant to the big goal and might well not be present in general. We now know that his original intuition was correct for he proved recently [18] that on a certain compact M^4 there are systems that can not be approximated by structurally stable ones. In [19] Smale gives many deep insights and makes several conjectures that are relevant for the fundamental problem.

(2.6) The fundamental problem can also be formulated in a similar way, if we restrict the class of fields $\mathfrak{X}$ under consideration to, say Hamiltonian systems, polynomial systems and so on. On the other hand one might also consider the same approach to more general problems, say the study of actions of R^p, $p > 1$, or a manifold, or the study of Pfaffian forms. Both are very difficult problems and the day seems to be far off when we will have of either of these situations a knowledge comparable to the one we have now of vector fields.

3. Structural stability.

On a compact differentiable manifold M^n a vector field $x \in \mathfrak{X}$ is said to be structurally stable if given $\varepsilon > 0$ one may find $\delta > 0$ such that whenever $\rho(X,Y) < \delta$ then $Y \sim X$ and the corresponding homeomorphism is within ε from the identity. Here ρ is a metric in $\mathfrak{X}$ and in M^n we assume that these is also a metric. One might give also a simpler definition involving no ε: X is structurally stable whenever there is $\delta > 0$ such that $\rho(X,Y) < \delta$ implies $X \sim Y$. In [11] the author proved that for

$n = 2$ these definition are equivalent. Calling $\Sigma \subset \mathfrak{X}$ the set of structurally stable systems, $n = 2$, $r \geq 1$, we proved also the following two facts

(3.1) $X \in \Sigma$ if and only if X satisfies:

a) singularities and closed orbits are generic

b) no trajectory connects two saddle points

c) the α- and ω-limit sets of any trajectory is either a singular point or a closed orbit.

(3.2) the subset of $\mathfrak{X}$ satisfying condition (3.1) is open and dense in $\mathfrak{X}$.

Since the system satisfying (3.1) exhibit a fairly simple structure the problem of classifying them into equivalence classes modulo homeomorphisms preserving trajectories offers no essential difficulty. The fundamental problem for compact M^2 can then be considered to be solved. The fact that it was done through the concept of structural stability gave some weight to this concept and it was natural to wonder whether or not in high dimensions Σ would be dense in $\mathfrak{X}$. Some indication in this direction was given by examples of Smale [20] and Anosov [4] exhibiting, for $n > 2$, structural stable systems with infinitely many closed orbits, a fact that can not happen if $n = 2$. As we mentioned above Smale proved recently that on a certain M^4, Σ is not dense in $\mathfrak{X}$ so that structural stability is too restictive a notion to be the answer to the fundamental problem.

Still, due to the obvious physical implication of this concept, it or perhaps some weaker version of it (relaxing the requirement of a homeomorphism mapping trajectories onto trajectories) seems to play a role in clarifying the qualitative theory of differential equations.

For one thing, progress in this theory seems to be essential in order to have a good basis for the so called "theory of bifurcation". Points of bifurcation are, in a space of parameters, points where the topological structure changes abruptly i.e. where structurally stability fails. A beginning in this direction is the work of J. Sotomayor [23] where for $n = 2$ he considers the structure of Banach manifold of codimension 1 that exists in a certain subset of $\mathfrak{X} - \Sigma$. This has lead him to characterize the arcs in $\mathfrak{X}$ which are in "general position with respect to Σ". Sotomayor needs $\mathfrak{X}$ with the C^r-topology, $r \geqq 3$.

Somehow related to this, in the sense that it gives some information about the geometry of Σ is the following theorem which we state here without proof.

(3.3) Theorem. If $n = 2$ and $X \in \Sigma$ then the fundamental group of Σ at X can be computed once we know the singularities and closed orbits of X; it is always finitely generated.

In dimension $n > 2$ very little is as yet known about structurally stable systems, the conjectures in [19] are still open. The following proposition seems to be true but at the present moment the author has no formal proof of it.

(3.4) If for $n > 2$ X is structurally stable then every minimal set μ of X which is not a singular point has dimension 1.

The reason for this is the following lemma whose proof offers no difficulty. Assume μ is a minimal set of dimension 1.

(3.5) Lemma. Given a point $p \in \mu$ one can find a flow-box $F \ni p$ (it can be made arbitrarily small) which is "transversed to μ".

By this it is meant that μ intersects ∂F only at the two faces which are transversal to the flow, keeping from the others at a distance which is bounded away from zero. This lemma implies that one can always find a cross-section about a point $p \in \mu$ where all the features of μ are present, as in the case of a closed orbit; the intersection of μ with the cross section is a Cantor set.

4. The General Density Theorem, the closing Lemma.

One basic step in the direction of the fundamental problem is to generalize (3.2) for dimension $n > 2$ i.e. to exhibit a number of generic properties of differential equations i.e. properties which are satisfies by a set $\mathcal{G} \subset \mathfrak{X}$ and which impose on the differential equations a certain amount of order and regularity. Hopefully after a reasonable number of these properties has been discovered one has an understanding of the main features of the differential equations involved, which will lead to a classification. At the present moment the best result on this line is the theorem below.

Before stating it we recall a concept due to Birkhoff. Let $\varphi_t : M \to M$ be the one-parameter group of diffeomorphisms generated by a vector field X on M. A point $p \in M$ is said to be non wandering if given any neighborhood U of p then there are arbitrarily large values of t for which $U \cap \varphi_t(U) \neq \emptyset$. Calling Ω the set of all non wandering points it is easy to see that Ω is compact, invariant through X and contains the α and ω limit sets of every trajectory of X.

(4.1) <u>General Density Theorem</u>. Let $\mathfrak{X}$ be the set of all vector fields on M^n, with the C^1-topology and let $\mathcal{G} \subset \mathfrak{X}$ be the subset of those for which the following G_i -properties are satisfied.

G_1: the singularities are generic, and so finite in number

G_2: the closed orbits are generic

G_3: the stable and unstable manifolds associated to the singularities and closed orbits are transversal.

G_4: $\Omega = \bar{\Gamma}$, where Γ stands for the union of all singular points and closed orbits of the vector field.

Then $\mathcal{G}$ is residual in $\mathfrak{X}$.

A subset of $\mathfrak{X}$ is residual if it contains a subset which is a countable intersection of subsets of $\mathfrak{X}$ which are open and dense in $\mathfrak{X}$. In particular $\mathcal{G}$ is dense in $\mathfrak{X}$ since $\mathfrak{X}$ is a Baire space.

The GDT as far as G_i, $i \leqq 3$, are concerned is due to Kupka and Smale, see [12] for a streamlined presentation. Then we have residuality even though we assume the C^r-topology, $r \geqq 1$, in $\mathfrak{X}$. The part concerning G_4 is due to C. Pugh [15,16] and it is responsible for the restriction $r = 1$. A weaker form of G_4, in this context, was conjectured by Smale [19].

For $n = 2$, a previous theorem of the author [11] gives a result stronger than the above GDT for then we know that $\mathcal{G}$ contains a set which is open and dense and in $\mathfrak{X}$ we may have the C^r-topology, $r \geqq 1$. Conversely, as has been pointed out by Pugh (see these Proceedings), if $n = 2$ the GDT plus a little extra work implies the above theorem for $r = 1$.

For $n > 2$ as a consequence of an example of Smale mentioned above [18] it follows that $\mathcal{G}$ does not contain a set open and dense in $\mathfrak{X}$. But it seems likely that the above GDT is also true for $r > 1$; if so, this would have a healthy effect of further developments of this theory. As mentioned before questions of bifurcation require $r \geqq 3$.

The restriction $r = 1$ comes from the fact that in order to show that the set of fields satisfying G_4 is residual one is faced with the problem of the closing Lemma. This, as generalized by Pugh, is as follows: given a non wandering point $p \in M^n$ of X, to find an arbitrarily C^r-small ΔX such that $X + \Delta X$ has a closed orbit through p. For $n = 2$ and in the special case where p is recurrent i.e. such that for every neighborhood U of p, $U \cap \varphi_t(p) \neq \emptyset$ for arbitrarily large values of t, this problem was already the crucial point in the theorem of the author mentioned above; the case $n > 2$, p recurrent was recognized in [11] as an important and difficult question. In [15] Pugh solves the closing Lemma for p recurrect, $n \geqq 2$, $r = 1$. In [16] he improves his result for p non wandering and gets G_4. By doing this, through his very difficult and ingenious proof, he made a fundamental contribution to the qualitative theory of differential question. But Pugh's proof is too long and complicated and one is left with the impression that the true methods to handle these questions are yet to be found.

5. The first integral theorem of Thom.

Let $M = M^n$ and $\mathfrak{X}$ be as before. A first integral of a vector field $X \in \mathfrak{X}$ is a differentiable function $f: M \to R$ which is constant along trajectories of X but is not constant on any open set of M. For technical reasons (Morse-Sard theorem) we consider only first integrals which are of class C^n. Traditionally, to "integrate" a differential equation is to find more and more first integrals. Of course they provide valuable information about the given equation when they can be found. In an unpublished manuscript R. Thom showed that the subset of $\mathfrak{X}$ of all fields which do not admit a first integral is residual, his proof being based on the assumption that the closing Lemma is true. Now we indicate how this theorem follows immediately from the GDT. For this we need only to show that $\mathcal{G}$ being as in (4.1) then we have the following.

(5.1) Theorem. If $X \in \mathcal{G}$ then X admits no first integral f.

Proof. Let X satisfy G_i, $i \leqq 4$, and let $f\colon M \to R$ be a first integral of X of class C^n. From the Morse-Sard theorem there is in $f(M)$ an interval (a,b) made up of regular values. For any $\alpha \in (a,b)$, $f^{-1}(\alpha)$ is an (n-1)-dimensional, compact, differentiable manifold, invariant under X. Now $f^{-1}(\alpha)$ contains no singularity or closed orbit of X because these are generic and in $f^{-1}(\alpha)$ there is no room for the corresponding stable and unstable manifolds. So the singularities and closed orbits of X are all located at the critical levels of f. Considering any trajectory γ in $f^{-1}(\alpha)$, $\omega(\gamma) \subset f^{-1}(\alpha)$ which is not contained in the closure of the set singularities and closed orbits, in contradiction with G_4. The theorem is proved.

The above argument actually shows also that

(5.2) no structurally stable system X in M admits a first integral f.

This follows from the fact that a structurally stable system satisfies G_4 and has only generic singularities and closed orbits, a fact easy to see . This is so whether we adopt the ε or non ε-difinition of structural stability, so that (5.2) is true in both cases. In the case of the ε-definition (5.2) was proved directly, without the use of the closing Lemma, by Arraut [4]. Clearly (5.1) and (5.2) remain true when f is only invariant through X i.e. f is constant on trajectories of X and is allowed to be constant on some open sets of M but not on the whole of M. The fact that no $X \in \mathcal{G}$ admits an invariant function throws some light on the global behavior of the stable and unstable manifolds showing that they are somehow tied together to each other and gives some indication in favor of a conjecture of Smale [19] that the union of them all is dense in M.

6. First integrals and structural stability: a conjecture.

As above let $M^n = M$ be compact, endowed with a Riemanian metric, and let $\Sigma \subset \mathfrak{X}$ be the set of all structurally stable systems on M.

Since $\overline{\Sigma} \neq \mathfrak{X}$ in dimension $n \geq 4$, it is natural to ask what lies in the closure of the structurally stable systems and in particular if the equations of conservative Dynamics, all of which have one first integral, are there. We assume that a first integral is a non degenerate Morse function and make the following conjecture.

(6.1) If $X \in \mathfrak{X}$ has a first integral f then it can be approximated by a structurally stable system i.e. $X \in \overline{\Sigma}$.

This conjecture can be considered as a generalization of the fact that a harmonic oscillator, which is not structurally stable, can be made so by the introduction of a small friction.

Below we indicate evidence for (6.1), actually we reduce

(6.1) to a known problem. Consider the following perturbation of X,

$$Y = X + \varepsilon \text{ grad } f$$

where $\varepsilon > 0$ is a small constant. Then the only singularities of Y are those which are common to X and grad f and so they are finite in number. Choose ε so that Y has only generic singularities. Then every trajectory of Y different from a singularity connects two singularities and along it f increases with time, as if Y were a gradient system. Now, using this fact and known techniques, see for instance [12], one can perturb Y to get a system Z such that every non singular trajectory again connects two generic singularities and besides all stable and unstable manifolds are transversal i.e. Z satisfies G_i, $i \leq 3$ without closed orbits. Systems of this type are usually said to be of Morse-Smale type.

The problem then reduces to show that Z is structurally stable. This is generally believed to be true, but there are technical difficulties. From what Smale says at the end of [17] our conjecture may be considered to be a fact if $n = 3$. What Thom says informally at the end of [24] practically implies that it is true in general, but his argument is not conclusive. A proof of (6.1) along the lines mentioned above would also be good to prove that on any compact manifold there are structurally stable systems, also a worthwhile result. We intend to come back to this question soon. The truth of (6.1) implies that the situation of [18], or any similar one, can never be present on a mechanical problem.

As a final comment on this we may add that on a mechanical situation where f is the energy function H the only equilibrium points of Z are among the critical points of H and the fact that they are generic (i.e. rigid) reminds us of the energy levels of quantum mechanics.

7. Further remarks on first integrals.

From what we saw above, on compact manifolds, the existence of a first integral imposes a "stratification" on the set of trajectories which turn out to be a very severe restriction and this is why, being incompatible with genericity, first integrals are irrelevant for the solution of the fundamental problem.

On a non compact M, see [12], the situation seems to be different though. Take for instance in R^2 the horizontal unit field $X = (1,0)$. If has a C^∞-first integral, is structurally stable, and very likely it is a generic field in any reasonable sense given to this expression. For a non compact M one should require that a first integral f, besides being a non degenerate Morse function, is such that the topological type of the inverse image does not change as long as we do not cross a critical value. This condition is satisfied when

f satisfies the condition (C) of Palais and Smale [21]: if S is a subset of M on which $|f|$ is bounded but on which grad f is not bounded away from zero then there is a critical point of f in $\overline{S}$.

Perhaps relevant along this line would be the consideration of systems X on M which are totally integrable i.e. which admit the maximum number of independent integrals $f_1,\ldots,f_{n-1}$; by independent one should mean independent except at some submanifold of dimension $< n-1$.

On a totally integrable system there is no room for complicated behavior of trajectories outside the critical levels and then one may pose the following problem: to characterize the totally integrable systems.

Even for the case of a compact M^n, $n > 2$, this might be an interesting question.

This problem suggests that one consider the relationship between singularities and first integrals. And then natural problems are: what singularities can belong to a totally integrable systems ?; given a generic singularity p such that not all eigenvalues have the real part with the same sign, how many independent first integrals do exist in a neighborhood of p ?

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